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Nice labeling problem for event structures: a counterexample

For Hans-Jürgen Bandelt on his 60th birthday

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Abstract. In this note, we present a counterexample to a conjecture of Rozoy and Thiagarajan from 1991 (called also the nice labeling problem) asserting that any (coherent) event structure with finite degree admits a labeling with a finite number of labels, or equivalently, that there exists a function $f:\mathbb{N}\mapsto\mathbb{N}$ such that an event structure with degree $\le n$ admits a labeling with at most $f(n)$ labels. Our counterexample is based on the Burling's construction from 1965 of 3-dimensional box hypergraphs with clique number 2 and arbitrarily large chromatic numbers and the bijection between domains of event structures and median graphs established by Barthélemy and Constantin in 1993.

1. Introduction

Event structures introduced by Nielsen, Plotkin, and Winskel [18, 25, 26] is a widely recognized abstract model of concurrent computation. An event structure is a partially ordered set of the occurrences of actions, called events, together with a conflict relation. The partial order captures the causal dependency of events. The conflict relation models incompatibility of events so that two events that are in conflict cannot simultaneously occur in any state of the computation. Consequently, two events that are neither ordered nor in conflict may occur concurrently. Formally, an *event structure* is a triple $\mathcal{E}=(E,\le,\smile)$, where

- E is a set of *events*,
- $\le\subseteq E\times E$ is a partial order of *causal dependency*,
- $\smile\subseteq E\times E$ is a binary, irreflexive, symmetric relation of *conflict*,
- $e\smile e'$ and $e'\le e''$ imply $e\smile e''$.

What we call here an event structure is usually called a coherent event structure or an event structure with a binary conflict. Additionally, the partial order $\le$ in the definition of an event structure is supposed to be *finitary,* i.e., the set $\{e'\in E: e'\le e\}$ is finite for any $e\in E$. Two events e',e'' are *concurrent* (notation $e'\frown e''$) if they are order-incomparable and they are not in conflict. Let e' and e'' be two elements in conflict. This conflict $e'\smile e''$ is said to be *minimal* if there is no element $e\ne e',e''$ such that either $e\le e'$ and $e\smile e''$ or

$e \leq e''$ and $e \smile e'$. Two elements are *independent* [3] (or *orthogonal* [22]) if they are either concurrent or in minimal conflict. An *independent* set is a subset of E whose elements are pairwise independent. The *degree* of an event structure $\mathcal{E}$ is the least upper bound of the sizes of the independent sets.

A labeling of an event structure $\mathcal{E}$ is a map λ from E to some alphabet Λ. The labeling λ is a *nice labeling* of $\mathcal{E}$ if any two independent events have different labels. Assous, Bouchitté, Charretton, and Rozoy [3] note that a nice labeling of an event structure "is equivalent to label the transitions by actions with the following condition: two transitions associated with the same initial state but with two different final states must have two different labels" and that the nice labeling conjecture of [20] formulated below arises when studying the equivalence of three different models of distributed computation: labeled event structures, transitions systems, and distributed monoids. Nice labeling of event structures was introduced by Rozoy and Thiagarajan [20] in their study of relationships between trace monoids and labeled event structures. Rozoy and Thiagarajan conjectured that *any event structure with finite degree admits a nice labeling with a finite number of labels.* In a quantitative version of this conjecture (which can be considered also for finite event structures) this conjecture can be re-formulated as: *there exists a function* $f : \mathbb{N} \mapsto \mathbb{N}$ *such that any event structure of degree* $\leq n$ *admits a nice labeling with at most* $f(n)$ *labels.*

Assous et al. [3] proved that the event structures of degree 2 admit nice labelings with 2 labels and noticed that Dilworth's theorem implies that the conflict-free event structures of degree n have nice labelings with n labels. They also proved that finding the least number of labels in a nice labeling of a finite event structure is NP-hard (by a reduction from graph coloring problem) and presented an example of a event structure of degree n requiring more than n labels. Recently, Santocanale [22] proved that all event structures of degree 3 and with tree-like partial orders have nice labelings with 3 labels. Both papers [3, 22] contain some other results and reformulations of the nice labeling problem. In particular, Santocanale [22] reformulated a nice labeling of an event structure $\mathcal{E}$ as a coloring problem of the *orthogonality graph* $\mathcal{G}(\mathcal{E})$ of $\mathcal{E}$: the vertices of $\mathcal{G}(\mathcal{E})$ are the events of $\mathcal{E}$ and two events are adjacent in $\mathcal{G}(\mathcal{E})$ if and only if they are orthogonal (i.e., independent). Then the independent sets of events become the cliques of $\mathcal{G}(\mathcal{E})$, the degree of $\mathcal{E}$ becomes the clique number of $\mathcal{G}(\mathcal{E})$, and the colorings of $\mathcal{G}(\mathcal{E})$ are in bijection with the nice labelings of $\mathcal{E}$.

In this note, we show that the conjecture of Rozoy and Thiagarajan is false already for event structures of degree 5. For this, we will use a more geometric and combinatorial view on event structures. Namely, we will use the bijections between domains of event structures and median graphs established by Barthélemy and Constantin [8] and between median graphs and CAT(0) cubical complexes established in [10, 19]. Together with those ingredients, our counterexample is based on the Burling's construction [9, 13] of 3-dimensional box hypergraphs with clique number 2 and arbitrarily large chromatic numbers.

2. Domains of event structures

In view of our geometric approach to event structures, it will be more convenient to reformulate and investigate the nice labeling conjecture in terms of domains, which we recall now. The domain of an event structure $\mathcal{E}$ consists of all computations states, called configurations. Each computation state is a subset of events subject to the constrains that no two conflicting events can occur together in the same computation and if an event occurred in a computation then all events on which it causally depends have occurred too. Formally, the set $\mathcal{D} = \mathcal{D}(\mathcal{E})$ of *configurations* of an event structure $\mathcal{E} = (E, \leq, \smile)$ consists of those subsets $C \subseteq E$ which are *conflict-free* ($e, e' \in C$ implies that e, e' are not in conflict) and *downward-closed* ($e \in C$ and $e' \leq e$ implies that $e' \in C$) [26]. The *domain* of an event structure is the set $\mathcal{D}(\mathcal{E})$ ordered by inclusion: if $C, C' \in \mathcal{D}(\mathcal{E})$ and $C' \subseteq C$, then C' can be viewed as a subbehaviour of C. Thus the partial order $\subseteq$ on $\mathcal{D}(\mathcal{E})$ expresses the progress in computation [26]. As is noticed in [22], (C', C) is a (directed) edge of the Hasse diagram of $(\mathcal{D}(\mathcal{E}), \subseteq)$ if and only if $C = C' \cup \{e\}$ for an event $e \in E \setminus C$, i.e., citing [26], "events manifest themselves as atomic jumps from one configuration to another". Then a nice labeling of the event structure $\mathcal{E}$ can be reformulated as a coloring of the directed edges of the Hasse diagram of its domain $\mathcal{D}(\mathcal{E})$ subject to the following local conditions [22]:

Determinism: transitions outgoing from the same state have different colors, i.e., the edges outgoing from the same vertex of $\mathcal{D}(\mathcal{E})$ have different colors;
Concurrency: the opposite edges of each square of the Hasse diagram of $\mathcal{D}(\mathcal{E})$ are colored in the same color.

As noticed in [22], there exists a bijection between such edge-colorings of the Hasse diagram of $\mathcal{D}(\mathcal{E})$ and nice labelings of $\mathcal{E}$ (i.e., colorings of the orthogonality graph $\mathcal{G}(\mathcal{E})$ of $\mathcal{E}$). Moreover, it is shown in Lemma 2.11 of [22] that $\{e_1, \ldots, e_n\}$ is a clique of $\mathcal{G}(\mathcal{E})$ if and only if there exists a configuration $C \in \mathcal{D}(\mathcal{E})$ such that for all $i = 1, \ldots, n$, $C \cup \{e_i\}$ are configurations and $(C, C \cup \{e_i\})$ are directed edges of the Hasse diagram of $\mathcal{D}(\mathcal{E})$. According to this result, the degree of an event structure $\mathcal{E}$ (alias the clique-number of the orthogonality graph of $\mathcal{E}$) equals to the maximum out-degree of a vertex in the Hasse diagram of $\mathcal{D}(\mathcal{E})$.

3. Domains, median graphs, and CAT(0) cubical complexes

We recall now the bijections between domains of event structures and median graphs established in [8] and between median graphs and 1-skeletons of CAT(0) cubical complexes established in [10, 19]. This will allow us to reformulate the nice labeling problem in truly geometric terms. Median graphs and related median structures (median algebras and CAT(0) cubical complexes) have many nice properties and admit numerous characterizations. These structures have been investigated in several contexts by quite a number of authors for more than half a century. We present here only a brief account of the characteristic properties of median structures; for more detailed information, the interested reader can consult the surveys [4, 6] and the book [24] (see also the papers [1, 11] in which CAT(0) cubical complexes are viewed as state complexes associated to metamorphic robots).

Let $G = (V, E)$ be simple, connected, without loops or multiple edges, but not necessarily finite graph. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path, and the *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)–paths, that is, of all vertices (metrically) *between* u and v:

$$I(u, v) := \{x \in V : d(u, x) + d(x, v) = d(u, v)\}.$$

An induced subgraph of G (or the corresponding vertex set) is called *convex* if it includes the interval of G between any of its vertices. A graph $G = (V, E)$ is *isometrically embeddable* into a graph $H = (W, F)$ if there exists a mapping $\varphi : V \to W$ such that $d_H(\varphi(u), \varphi(v)) = d_G(u, v)$ for all vertices $u, v \in V$.

A graph G is called *median* if the interval intersection $I(x, y) \cap I(y, z) \cap I(z, x)$ is a singleton for each triplet x, y, z of vertices. Median graphs are bipartite. Basic examples of median graphs are trees (which are successive point amalgams of K_2), hypercubes (which are Cartesian powers of K_2), rectangular grids (which are Cartesian products of finite or infinite paths), and covering graphs of distributive lattices. With any vertex v of a median graph $G = (V, E)$ is associated a canonical partial order $\leq_v$ defined by setting $x \leq_v y$ if and only if $x \in I(v, y)$; v is called the basepoint of $\leq_v$. Since G is bipartite, the Hasse diagram G_v of the partial order $(V, \leq_v)$ is the graph G in which any edge xy is directed from x to y if and only if the inequality $d(x, v) < d(y, v)$ holds. We call G_v a *pointed median graph.* Theorems 2.2 and 2.3 of Barthélemy and Constantin [8] establish the following bijection between event structures and pointed median graphs (in [8], event structures are called sites):

Theorem 1. [8] *The (undirected) covering graph of the domain $(\mathcal{D}(\mathcal{E}), \subseteq)$ of any event structure $\mathcal{E} = (E, \leq, \smile)$ is a median graph. Conversely, for any median graph G and any basepoint v of G, the pointed median graph G_v is isomorphic to the Hasse diagram of a domain of an event structure.*

We only recall how to define the event structure occurring in the second part of this theorem. For this, we will introduce some notions which will be also used in the description of our counterexample. Median graphs are isometric subgraphs of hypercubes and Cartesian product of trees [7, 17]. The isometric embedding of a median graph G into a (smallest) hypercube coincides with the so-called canonical embedding, which is determined by the Djoković-Winkler relation Θ on the edge set of G : two edges xy and zw are Θ-related exactly when

$$d_G(x, z) + d_G(y, w) \neq d_G(x, w) + d_G(y, z).$$

For a median graph this relation is transitive and hence an equivalence relation. It is the transitive closure of the “opposite” relation of edges on 4-cycles: in fact, any two Θ-related edges can be connected by a ladder (viz., the Cartesian product of a path with K_2), and the block of all edges Θ-related to some edge xy constitute a cutset $\Theta(xy)$ of the median graph, which determines one factor of the canonical hypercube [16, 17]. The cutset $\Theta(xy)$ defines a convex split $\sigma(xy) = \{W(x, y), W(y, x)\}$ of G [17], where $W(x, y) = \{z \in V : d(z, x) < d(z, y)\}$ and $W(y, x) = V - W(x, y)$ (we will call the complementary convex sets $W(x, y)$

and $W(y,x)$ *halfspaces*; they are not only convex but also gated, see the definition below). Conversely, for every convex split of a median graph G there exists at least one edge xy such that $\{W(x,y), W(y,x)\}$ is the given split. We will denote by $\{\Theta_i : i \in I\}$ the equivalence classes of the relation Θ (in [8], they were called parallelism classes).

Suppose that v is an arbitrary but fixed basepoint of a median graph G. For an equivalence class $\Theta_i, i \in I$, we will denote by $\sigma_i = \{A_i, B_i\}$ the associated convex split, and suppose without loss of generality that $v \in A_i$. Two equivalence classes Θ_i and Θ_j are said to be *incompatible* or *crossing* if there exists a 4-cycle C of G with two opposite edges in Θ_i and two other opposite edges in Θ_j (Θ_i and Θ_j are called *compatible* otherwise). An equivalence class Θ_i *separates* the basepoint v from the equivalence class Θ_j if Θ_i and Θ_j are compatible and all edges of Θ_j belong to B_i. The event structure $\mathcal{E}_v = (E, \leq, \smile)$ associated with a pointed median graph G_v is defined in the following way. E is the set $\{\Theta_i, i \in I\}$ of the equivalence classes of Θ. The causal dependency is defined by setting $\Theta_i \leq \Theta_j$ if and only if $\Theta_i = \Theta_j$ or Θ_i separates v from Θ_j. Finally, the conflict relation is defined by setting $\Theta_i \smile \Theta_j$ if and only if Θ_i and Θ_j are compatible, Θ_i does not separate v from Θ_j and Θ_j does not separates v from Θ_i. Theorem 2.3 of [8] shows that G_v is indeed the Hasse diagram of the domain $\mathcal{D}(\mathcal{E}_v)$ of the event structure $\mathcal{E}_v$.

In our counterexample we will use the following constructive characterization of finite median graphs. A subset W of V or the subgraph H of $G = (V, E)$ induced by W is called *gated* (in G) if for every vertex x outside H there exists a vertex x' (the *gate* of x) in H such that each vertex y of H is connected with x by a shortest path passing through the gate x'. In general, any gated set is convex. In median graphs, all convex sets are gated. A graph G is a *gated amalgam* of two graphs G_1 and G_2 if G_1 and G_2 constitute two intersecting gated subgraphs of G whose union is all of G. Equivalently, G is a gated amalgam of G_1 and G_2 if the intersection G_0 of G_1 and G_2 in G is a gated subgraph of G_1 and G_2 and G_0 separates in G any vertex of $G_1 \setminus G_2$ from any vertex of $G_2 \setminus G_1$.

Theorem 2. [15, 23] *The gated amalgam of two median graphs is a median graph. Moreover, every finite median graph G can be obtained by successive applications of gated amalgamations from hypercubes.*

Now we recall the close relationship between the median graphs and CAT(0) cubical complexes. We believe that it is worth putting together event structures, median graphs, and CAT(0) cubical complexes because some problems similar to the nice labeling problem have been independently formulated in geometric and combinatorial settings. Notice also that a result similar to the result of Barthélemy and Constantin [8] has been rediscovered recently in [2] in the context of CAT(0) cubical complexes. Finally, it may happen that combinatorial, structural, algebraic, geometrical, and group theoretical results established for median structures or CAT(0) cubical complexes can be useful for the investigation of event structures.

A *cubical complex* $\mathcal{K}$ is a set of solid cubes of any dimensions which is closed under taking subcubes and nonempty intersections. For a complex $\mathcal{K}$ denote by $V(\mathcal{K})$ and $E(\mathcal{K})$ the *vertex set* and the *edge set* of $\mathcal{K}$, namely, the set of all 0-dimensional and 1-dimensional cubes of

$\mathcal{K}$. The pair $G(\mathcal{K}) = (V(\mathcal{K}), E(\mathcal{K}))$ is called the *(underlying) graph* or the *1-skeleton* of $\mathcal{K}$. Conversely, for a graph G one can derive a cubical complex $\mathcal{K}(G)$ by replacing all graphic cubes of G by solid cubes. The cubical complex $\mathcal{K}(G)$ associated with a median graph G is called a *median cubical complex*. If instead of a solid cube, we replace each graphic cube of G by an axis-parallel box (i.e., if instead of length 1 we take length l_i for all edges from the same equivalence class Θ_i), then we will get a *median box complex*. Median cubical and median box complexes endowed with the intrinsic l_1-metric are median metric spaces (i.e., every triplet of points has a unique median) and therefore are l_1-subspaces [24]. Finally, if we impose the intrinsic l_2-metric on a median cubical or box complex, then we obtain a metric space with global non-positive curvature.

A *geodesic triangle* $\Delta = \Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic between each pair of vertices (the sides of Δ). A *comparison triangle* for $\Delta(x_1, x_2, x_3)$ is a triangle $\Delta(x'_1, x'_2, x'_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(x'_i, x'_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space (X, d) is defined to be a *CAT(0) space* [12] if all geodesic triangles $\Delta(x_1, x_2, x_3)$ of X satisfy the comparison axiom of Cartan–Alexandrov–Toponogov: *If y is a point on the side of $\Delta(x_1, x_2, x_3)$ with vertices x_1 and x_2 and y' is the unique point on the line segment $[x'_1, x'_2]$ of the comparison triangle $\Delta(x'_1, x'_2, x'_3)$ such that $d_{\mathbb{E}^2}(x'_i, y') = d(x_i, y)$ for $i = 1, 2$, then $d(x_3, y) \leq d_{\mathbb{E}^2}(x'_3, y')$.* CAT(0) spaces can be characterized in several different natural ways, in particular, a geodesic metric space (X, d) is CAT(0) if and only if any two points of this space can be joined by a unique geodesic. Several classes of CAT(0) complexes can be characterized combinatorially, and the characterization of cubical CAT(0) complexes given by M. Gromov is especially nice:

Theorem 3. [12] *A cubical or box complex $\mathcal{K}$ with the l_2-metric is CAT(0) if and only if $\mathcal{K}$ is simply connected and whenever three $(k+2)$-cubes of $\mathcal{K}$ share a common k-cube and pairwise share common $(k+1)$-cubes, they are contained in a $(k+3)$–cube of $\mathcal{K}$.*

The following relationship holds between CAT(0) cubical complexes and median cubical complexes.

Theorem 4. [10, 19] *Median cubical (box) complexes and CAT(0) cubical (box) complexes (both equipped with the l_2-metric) constitute the same objects.*

The proof of this theorem given in [10] is self-contained and allows to derive some properties of CAT(0) cubical complexes from known results about median graphs. In particular, a fundamental result of Sageev [22] that each hyperplane of a CAT(0) cubical complex $\mathcal{K}$ does not self-intersect and partition $\mathcal{K}$ in exactly two parts is a consequence of the fact that each equivalence class Θ_i of the median graph $G(\mathcal{K})$ defines a convex split $\{A_i, B_i\}$. A *hyperplane* H_i associated to Θ_i is the cubical complex whose 1-skeleton is the graph in which the middles (baricenters) of the edges of Θ_i are the vertices and two such vertices are adjacent if they are middles of two opposite edges of a square of $\mathcal{K}$. The *carrier* $N(H_i)$ in $\mathcal{K}$ of a hyperplane H_i is the union of all cubes of $\mathcal{K}$ crossed by H_i, i.e. the union of all cubes having an edge in the equivalence class Θ_i. In [14], Hagen introduced and investigated in depth the important

concept of a contact graph of a CAT(0) cubical complex $\mathcal{K}$. According to [14], the contact graph $\Gamma = \Gamma(\mathcal{K}) = \Gamma(G(\mathcal{K}))$ of $\mathcal{K}$ is a graph having the hyperplanes (or the equivalence classes of Θ) as vertices and two hyperplanes H_i and H_j are adjacent in Γ (notation $H_i \perp\!\!\!\!\!\!- H_j$) if and only if the carriers $N(H_i)$ and $N(H_j)$ intersect. It was noticed in [14] that if $H_i \perp\!\!\!\!\!\!- H_j$, then the hyperplanes H_i and H_j either *cross* (in which case the equivalence classes Θ_i and Θ_j cross) or *osculate* (in which case there exist two edges $e \in \Theta_i$ and $e' \in \Theta_j$ sharing a common endpoint and not belonging to a common square). Analogously to the equality between the clique-number of the orthogonality graph of an event structure and the degree of the event structure, the clique number $\omega(\Gamma)$ of the contact graph equals to the maximum degree of the graph $G(\mathcal{K})$ of $\mathcal{K}$.

Due to the established bijections between event structures and their domains, between domains and pointed median graphs, and between median graphs and CAT(0) cubical complexes, one can view the orthogonality graph of an event structure and the crossing graph of a median graph as subgraphs of the contact graph of the associated CAT(0) cubical complex. Namely, the *crossing graph* $\Gamma_{\#} = \Gamma_{\#}(G) = \Gamma_{\#}(\mathcal{K}(G))$ of a median graph G (or of the associated cubical complex $\mathcal{K}(G)$) has the hyperplanes of $\mathcal{K}(G)$ (or the equivalence classes of Θ) as vertices and the pairs of crossing hyperplanes as edges. Now, let G_v be a pointed median graph obtained from G. As we noticed already (and this follows easily from the definition of the halfspaces), all edges of any equivalence class Θ_i of G are directed in G_v from A_i to B_i, i.e., if $xy \in \Theta_i$ and $v, x \in A_i, y \in B_i$, then xy is directed from x to y. The *pointed contact graph* $\Gamma_v = \Gamma_v(G) = \Gamma_v(\mathcal{K}(G))$ of G has the set of hyperplanes of $\mathcal{K}(G)$ (or the equivalence classes of Θ) as vertices and two hyperplanes H_i and H_j are adjacent if and only if either they cross or they osculate in two directed edges $e \in \Theta_i$ and $e' \in \Theta_j$ with a common origin (one can view Γ_v as the orthogonality graph of the event structure having the pointed median graph G_v as a domain). Since the relation Θ is transitive, in any coloring of the edges of the pointed median graph G_v satisfying the determinism and concurrency conditions, all edges of an equivalence class Θ_i have the same color, two crossing equivalence classes have different colors, and two edges with common origin have different colors. Hence, the colorings of edges of G_v are in bijection with the colorings of the pointed contact graph Γ_v. On the other hand, if we color the equivalence classes of Θ so that this is a coloring of the crossing graph $\Gamma_{\#}$ of G, then this corresponds to a coloring of edges of G using the concurrency rule only: the edges of each square of G are colored in two colors with opposite edges having the same color. By Proposition 1 of [5], coloring $\Gamma_{\#}$ in n colors is equivalent to an isometric embedding of G into the Cartesian product of n trees.

One can ask whether the *chromatic number of each of the graphs* $\Gamma, \Gamma_{\#}$, *or* Γ_v *is bounded by a function of its clique number*. In case of the pointed contact graph Γ_v, this is exactly the nice labeling problem. In case of the contact graph Γ, this question was raised by Hagen in the first version of [14]. On the other hand, it is well-known [7] (see also Proposition 2.17 of [14]) that any graph can be realized as the crossing graph of a median graph. Since there exists triangle-free graphs with arbitrarily high chromatic numbers, the chromatic number of $\Gamma_{\#}$ cannot be bounded by its clique number. Nevertheless, M. Sageev (personal communication from M.

Hagen) and, independently, the author of the present note asked whether the *chromatic number of* $\Gamma_{\#} = \Gamma_{\#}(G)$ *is bounded by a function of the maximum degree of the median graph* G*, i.e., if a median graph* G *with bounded degrees of vertices can be isometrically embedded into a bounded number of trees.*. In the next section, we will answer in the negative the first question about the graphs Γ and Γ_v. In a forthcoming paper with Hagen, we will modify this example to answer in the negative the last question about $\Gamma_{\#}$.

4. The counterexample

Our counterexample to the labeling conjecture of [20] is based on examples of Burling of box hypergraphs with clique number 2 and arbitrarily large chromatic numbers. Parallelepipeds in $\mathbb{R}^3$ whose sides are parallel to the coordinate axes are called *3-dimensional boxes*. Given a (finite) collection $\mathcal{B}$ of 3-dimensional boxes, its clique number $\omega(\mathcal{B})$ is the maximum number of pairwise intersecting boxes of $\mathcal{B}$ and its chromatic number $\chi(\mathcal{B})$ is the minimum number of colors into which we can color the boxes of $\mathcal{B}$ in such a way that any pair of intersecting boxes is colored in different colors (i.e., $\chi(\mathcal{B})$ is the chromatic number of the intersection graph of the family $\mathcal{B}$). In his PhD thesis [9], for each integer $n > 0$, Burling constructed a collection of axis-parallel boxes $\mathcal{B}(n)$ with clique number $\omega(\mathcal{B}(n)) = 2$ and chromatic number $\chi(\mathcal{B}(n)) > n$ (a full description of this construction is available in the survey paper by Gyárfás [13]).

Let B_0 be a box of $\mathbb{R}^3$. Suppose without loss of generality that one corner of B_0 is the origin of coordinates of $\mathbb{R}^3$ and that B_0 is located in the first octant of $\mathbb{R}^3$. Suppose that B_0 is subdivided into smaller boxes (called *elementary cells*) using a family of planes parallel to the three coordinate hyperplanes. This subdivision of B_0 defines a box complex as well as a cubical complex (if we scale all length of edges of the resulting boxes to 1). We denote both these complexes by $\mathcal{K}$ and by $G = G(\mathcal{K})$ their 1-skeleton. Notice that if B_0 is subdivided by $k_1 - 2$ planes parallel to the xy-plane, $k_2 - 2$ planes parallel to the yz-plane, and $k_3 - 2$ planes parallel to the xz-plane, then G is isomorphic to the $k_1 \times k_2 \times k_3$ grid, and therefore is a median graph ($\mathcal{K}$ is a CAT(0) cubical complex because its underlying space is the box B_0).

Let $\mathcal{B} = \{B_1, \dots, B_m\}$ be a box hypergraph such that the eight corners of each box $B_i, i = 1, \dots, m$, are vertices of the grid G. In this case, we say that the box hypergraph $\mathcal{B}$ is *cell-represented* by the box complex $\mathcal{K}$ because each box B_i is the union of elementary cells of $\mathcal{K}$. Let $\mathcal{K}_i$ be the subcomplex of $\mathcal{K}$ consisting of all elementary cells included in B_i and let $G_i = G(\mathcal{K}_i)$ be its underlying graph. Note that G_i is also a 3-dimensional grid. Hence G_i is a convex (and therefore gated) subgraph of G.

Now, we define a *lifting procedure* taking as an input the box complex $\mathcal{K}$, its graph G, and a box hypergraph $\mathcal{B}$ cell-represented by $\mathcal{K}$, and giving rise to a 4-dimensional CAT(0) box complex $\widetilde{\mathcal{K}}$ and its underlying median graph $\widetilde{G} = G(\widetilde{\mathcal{K}})$. The complex $\widetilde{\mathcal{K}}$ is realized in the $(m+3)$-dimensional space $\mathbb{R}^{m+3}$. Suppose that the 3-dimensional space in which we defined the box complex $\mathcal{K}$ is the subspace of $\mathbb{R}^{m+3}$ defined by the last 3 coordinates, i.e., each point p of B_0 has the coordinates $(0, \dots, 0, p_{m+1}, p_{m+2}, p_{m+3})$ with $p_{m+1}, p_{m+2}, p_{m+3} \geq 0$. Then for each box $B_i, i = 1, \dots, m$, we can define the numbers $0 \leq a'_i < a''_i, 0 \leq b'_i < b''_i, 0 \leq c'_i < c''_i$ so that B_i is the set of all points $p = (0, \dots, 0, p_{m+1}, p_{m+2}, p_{m+3}) \in \mathbb{R}^{m+3}$ such that

$p_{m+1} \in [a_i', a_i'']$, $p_{m+2} \in [b_i', b_i'']$, and $p_{m+3} \in [c_i', c_i'']$. Let $\widetilde{B}_i$ be the 4-dimensional box which is the Cartesian product of B_i with the unit segment s_i of the ith coordinate-axis of $\mathbb{R}^{m+3}$: $\widetilde{B}_i$ consists of all points $p = (p_1, \ldots, p_m, p_{m+1}, p_{m+2}, p_{m+3}) \in \mathbb{R}^{m+3}$ such that $p_j = 0$ for each $1 \leq j \leq m, j \neq i$, $p_i \in [0,1]$, $p_{m+1} \in [a_i', a_i''], p_{m+2} \in [b_i', b_i'']$, and $p_{m+3} \in [c_i', c_i'']$. Let $\widetilde{\mathcal{B}} = \{\widetilde{B}_i : B_i \in \mathcal{B}\}$ be the resulting box hypergraph in $\mathbb{R}^{m+3}$. Each elementary cell C of $\mathcal{K}$ gives rise to a 4-dimensional box $\widetilde{C}_i = C \times s_i$ for each 3-dimensional box $B_i \in \mathcal{B}$ containing C. We refer to each $\widetilde{C}_i$ as a *lifted elementary box*. Denote by $\widetilde{\mathcal{K}}$ the box complex consisting of all elementary cells of $\mathcal{K}$ and of all lifted elementary cells. For each box $\widetilde{B}_i$, let $\widetilde{\mathcal{K}}_i$ be the subcomplex of $\widetilde{\mathcal{K}}$ consisting of all lifted elementary cells $\widetilde{C}_i$ such that $C \subseteq B_i$. Finally, let $\widetilde{G} = G(\widetilde{\mathcal{K}})$ and $\widetilde{G}_i = G(\widetilde{\mathcal{K}}_i)$ be the 1-skeletons of $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{K}}_i$, respectively. Notice also that each $\widetilde{G}_i$ is a 4-dimensional rectangular grid because $\widetilde{G}_i$ is the Cartesian product $\widetilde{G}_i = G_i \times e_i$ of the grid G_i with an edge e_i. Notice that the edges $e_1, \ldots, e_m$ are different because the unit segments $s_1, \ldots, s_m$ belong to different coordinate-axes of $\mathbb{R}^{m+3}$.

Lemma 1. *$\widetilde{G}$ is a median graph.*

Proof. We will show that $\widetilde{G}$ is obtained from the median graphs $G, \widetilde{G}_1, \ldots, \widetilde{G}_{m-1}, \widetilde{G}_m$ by applying successive gated amalgamations. We proceed by induction on the size of the box hypergraph $\mathcal{B}$. If $\mathcal{B} = \varnothing$, then the result is immediate because $\widetilde{G}$ coincide with G. Now suppose by induction assumption that our assertion holds for the box hypergraph $\mathcal{B}' = \{B_1, \ldots, B_{m-1}\}$. Let $\widetilde{G}'$ be the median graph which is the 1-skeleton of the box complex $\widetilde{\mathcal{K}}'$ defined for the box hypergraph $\widetilde{\mathcal{B}}' = \{\widetilde{B}_1, \ldots, \widetilde{B}_{m-1}\}$. Notice that $\widetilde{G}$ is obtained by amalgamating the median graphs $\widetilde{G}'$ and $\widetilde{G}_i$ along their common subgraph G_i. Since G_i is a grid, G_i is a gated subgraph of the grid $\widetilde{G}_i$. Analogously, since G_i is a gated subgraph of G and the grid G is a gated subgraph of each of the median graphs resulting from previous gated amalgams, G_i is a gated subgraph of $\widetilde{G}'$. Thus $\widetilde{G}$ is a gated amalgam of two median graphs $\widetilde{G}_i$ and $\widetilde{G}'$, whence $\widetilde{G}$ is a median graph by Theorem 2. $\square$

Let $\Theta_i, i = 1, \ldots, m$, be the equivalence class of the edges of the median graph $\widetilde{G}$ defined by e_i, i.e., Θ_i consists of all edges of $\widetilde{G}$ (or of $\widetilde{\mathcal{K}}$) which are parallel to the ith coordinate-axis of $\mathbb{R}^{m+3}$. Each edge of Θ_i has one endpoint in the grid G_i of the box B_i and, vice versa, each vertex of G_i is an endpoint of exactly one edge of Θ_i. Θ comprises also other equivalence classes, namely the equivalence classes of the grid G augmented by the edges of $\widetilde{G}_1, \ldots, \widetilde{G}_m$ parallel to them.

Let α be the corner of B_0 which is identified with the origin of coordinates of $\mathbb{R}^{m+3}$ and let $\widetilde{G}_\alpha$ be the median graph $\widetilde{G}$ pointed at α. Then each edge e of the equivalence class Θ_i will be directed in $\widetilde{G}_\alpha$ away from G_i, i.e., the endpoint of e from G_i will be the origin of e. The edges $e = uv$ of the remaining equivalence classes will be directed in $\widetilde{G}_\alpha$ from a vertex with smaller coordinates to a vertex with larger coordinates (notice that, if fact, u and v will differ in exactly one of the last three coordinates). Let $\Gamma_\alpha(\widetilde{G})$ be the pointed contact graph of $\widetilde{G}_\alpha$.

Lemma 2. *Any two equivalence classes Θ_i and Θ_j do not cross.*

Proof. The hyperplane H_i of the CAT(0) box complex $\widetilde{\mathcal{K}}$ defined by Θ_i lies in the (3-dimensional) plane Π_i of $\mathbb{R}^{m+3}$ described by the equations $x_k = 0$ if $1 \le k \le m, k \neq i$, and $x_i = \frac{1}{2}$. Analogously, the hyperplane H_j defined by Θ_j lies in the plane Π_j of $\mathbb{R}^{m+3}$ described by the equations $x_k = 0$ if $1 \le k \le m, k \neq j$, and $x_j = \frac{1}{2}$. Since Π_i and Π_j are disjoint, the hyperplanes H_i and H_j are disjoint as well, thus the equivalence classes Θ_i and Θ_j do not cross. □

Lemma 3. *Two equivalence classes Θ_i and Θ_j are adjacent in $\Gamma_\alpha(\widetilde{G})$ if and only if the grids G_i and G_j intersect and if and only if the boxes B_i and B_j intersect.*

Proof. First suppose that Θ_i and Θ_j are adjacent in $\Gamma_\alpha(\widetilde{G})$. By Lemma 2 and the definition of edges of $\Gamma_\alpha(\widetilde{G})$, there exist two directed edges $e' \in \Theta_i$ and $e'' \in \Theta_j$ having the same origin. Since the origin of e' belongs to G_i and the origin of e'' belongs to G_j, we conclude that this common origin belongs to $G_i \cap G_j$ and therefore to $B_i \cap B_j$.

Conversely, suppose that B_i and B_j intersect. Since each of B_i and B_j is constituted by elementary cells of $\mathcal{K}$, we conclude that there exist two elementary cells $C' \subseteq B_i$ and $C'' \subseteq B_j$ which intersect. Necessarily C' and C'' share a common vertex v of G. Hence v is a vertex of both grids G_i and G_j. Now, from our construction follows that v is the origin of an edge e' of Θ_i and of an edge e'' of Θ_j, whence the equivalence classes Θ_i and Θ_j are adjacent in $\Gamma_\alpha(\widetilde{G})$. □

Lemma 4. *If the clique number of the box hypergraph $\mathcal{B}$ is $\omega = \omega(\mathcal{B})$, then the out-degree of a vertex in the pointed median graph $\widetilde{G}_\alpha$ is $\omega + 3$. In particular, the maximum degree of $\widetilde{G}$ is at most $\omega + 6$ and the clique number of the pointed contact graph $\Gamma_\alpha(\widetilde{G})$ is $\omega + 3$.*

Proof. Consider a maximal by inclusion collection $\mathcal{B}_0$ of k pairwise intersecting boxes of $\mathcal{B}$. By Lemma 3, for any two boxes $B_i, B_j \in \mathcal{B}_0$, the grids G_i and G_j intersect. From Helly property for convex sets of median graphs, we conclude that all grids G_i with $B_i \in \mathcal{B}_0$ share a common vertex v. The out-degree of v in the pointed median graph $\widetilde{G}_\alpha$ is at most $k+3$ because v is the origin of one edge from each of the k equivalence classes Θ_i with $B_i \in \mathcal{B}_0$ as well as the origin of at most three outgoing edges in the pointed grid G_α. Since $k \le \omega$, the out-degree of v is at most $\omega + 3$ showing that the out-degree in $\widetilde{G}_\alpha$ of any vertex of the grid G is at most $\omega + 3$. On the other hand, the out-degree of each vertex z of $\widetilde{G}$ not belonging to G is equal to the out-degree in G_α of its twin in G and therefore is at most 3. Since any vertex of $\widetilde{G}$ can have at most three incoming edges, we conclude that the maximum degree of a vertex of the undirected graph $\widetilde{G}$ is at most $\omega + 6$. □

Lemma 5. $\chi(\Gamma(\widetilde{G})) \ge \chi(\Gamma_\alpha(\widetilde{G})) \ge \chi(\mathcal{B})$.

Proof. The first inequality is obvious because $\Gamma_\alpha(\widetilde{G})$ is a subgraph of $\Gamma(\widetilde{G})$. By Lemma 3, any coloring of $\Gamma(\widetilde{G})$ restricted to the equivalence classes $\Theta_i, i = 1, \ldots, m$, provides a coloring of the box hypergraph $\mathcal{B}$. □

Now we will apply our construction to Burling's examples. For each integer $n > 0$, let $\mathcal{B}(n)$ be a box hypergraph with clique number 2 and chromatic number $\chi(\mathcal{B}(n)) > n$ as

defined in [9, 13]. Suppose that $\mathcal{B}(n)$ is drawn in the first open octant of $\mathbb{R}^3$. Let $B_0(n)$ be an additional axis-parallel box having one corner in the origin $\alpha(n)$ of coordinates of $\mathbb{R}^3$ and containing all boxes of $\mathcal{B}(n)$ ($B_0(n)$ will play the role of the box B_0). Let $\beta(n)$ be the corner of $B_0(n)$ opposite to $\alpha(n)$. Now, subdivide $B_0(n)$ into elementary cells by drawing the three axis-parallel planes trough each of eight corners of each box B_i of $\mathcal{B}(n)$. Denote the resulting grid by $G(n)$ and the resulting box complex (subdividing $B_0(n)$) by $\mathcal{K}(n)$. Then the box hypergraph $\mathcal{B}(n)$ is cell-represented by $\mathcal{K}(n)$. Denote by $\widetilde{\mathcal{K}}(n)$ the box complex obtained by applying our lifting procedure to $\mathcal{K}(n)$ and $\mathcal{B}(n)$. Let $\widetilde{G}(n) = G(\widetilde{\mathcal{K}}(n))$ be the 1-skeleton of $\widetilde{\mathcal{K}}(n)$ and let $\widetilde{G}_{\alpha(n)}(n)$ be the median graph $\widetilde{G}(n)$ pointed at $\alpha(n)$. Since $\omega(\mathcal{B}(n)) = 2$, from Lemma 4 we conclude that the maximum out-degree of a vertex in the pointed median graph $\widetilde{G}_{\alpha_n}(n)$ is at most 5. On the other hand, since $\chi(\mathcal{B}(n)) > n$, from Lemma 5 we conclude that the chromatic numbers of the contact graph of $\widetilde{G}(n)$ and of the pointed contact graph of $\widetilde{G}_{\alpha_n}(n)$ are larger than n. Summarizing, we obtain the following conclusion:

Proposition 1. *For any $n > 0$, there exist a median graph $\widetilde{G}(n)$ of maximum degree 8 and a pointed median graph $\widetilde{G}_{\alpha_n}(n)$ of maximum out-degree 5 such that any coloring of the contact graph of $\widetilde{G}(n)$ and of the pointed contact graph of $\widetilde{G}_{\alpha_n}(n)$ requires more than n colors. In particular, any nice labeling of the event structure $\mathcal{E}_{\alpha(n)}$ (of degree 5) whose domain is $\widetilde{G}_{\alpha_n}(n)$ requires more than n labels.*

To present a counterexample to the conjecture of Rozoy and Thiagarajan, we consider the following infinite median graph $\widetilde{G}^*$ whose blocks (2-connected components) are the graphs $\widetilde{G}(1), \widetilde{G}(2), \ldots$. Recall that each graph $\widetilde{G}(n)$ has two distinguished vertices $\alpha(n)$ and $\beta(n)$ which are opposite corners of the box $B_0(n)$. To construct $\widetilde{G}^*$, for each $n > 1$, we identify the vertex $\beta(n-1)$ of $\widetilde{G}(n-1)$ with the vertex $\alpha(n)$ of $\widetilde{G}(n)$ and obtain an infinite in one direction chain of blocks. Notice that the identified vertices $\beta(n-1) = \alpha(n)$ are exactly the articulation vertices of $\widetilde{G}^*$. Obviously $\widetilde{G}^*$ is a median graph because each its block is median. Now suppose that $\widetilde{G}^*$ is pointed at the vertex $\alpha = \alpha(1)$ and let $\widetilde{G}^*_\alpha$ be the resulting pointed median graph. Notice that each edge e of $\widetilde{G}^*$ is oriented in $\widetilde{G}^*_\alpha$ in the same way as in the orientation $\widetilde{G}_{\alpha_n}(n)$ of the unique block $\widetilde{G}(n)$ containing e. On the other hand, the out-degree in $\widetilde{G}^*_\alpha$ of each vertex v belonging to a unique block $\widetilde{G}(n)$ is the same as in $\widetilde{G}_{\alpha_n}(n)$ while the out-degree of each articulation point is 3, whence the maximum out-degree of $\widetilde{G}^*_\alpha$ is also 5. The pointed contact graph $\Gamma(\widetilde{G}^*_\alpha)$ of $\widetilde{G}^*_\alpha$ is the disjoint union of the pointed contact graphs $\Gamma(\widetilde{G}_{\alpha_n}(n))$. From Proposition 1 we conclude that the chromatic number of $\Gamma(\widetilde{G}^*_\alpha)$ is infinite. As a consequence, we established the following main result of this note:

Theorem 5. *There exists a pointed median graph $\widetilde{G}^*_\alpha$ of maximum out-degree 5 such that the chromatic number of its pointed contact graph $\Gamma(\widetilde{G}^*_\alpha)$ is infinite. In particular, any nice labeling of the event structure $\mathcal{E}_\alpha$ (of degree 5) whose domain is $\widetilde{G}^*_\alpha$ requires an infinite number of labels.*

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